

The Inconsistency of the Usual Galilean Transformation in Quantum Mechanics and How To Fix It

Daniel M. Greenberger

City College of New York, New York, NY 10031, USA

Reprint requests to Prof. D. G.; dansuzy@idt.net

Z. Naturforsch. **56a**, 67–75 (2001); received February 8, 2001

Presented at the 3rd Workshop on Mysteries, Puzzles and Paradoxes in Quantum Mechanics, Gargnano, Italy, September 17–23, 2000

It is shown that the generally accepted statement that one cannot superpose states of different mass in non-relativistic quantum mechanics is inconsistent. It is pointed out that the extra phase induced in a moving system, which was previously thought to be unphysical, is merely the non-relativistic residue of the “twin-paradox” effect. In general, there are phase effects due to proper time differences between moving frames that do not vanish non-relativistically. There are also effects due to the equivalence of mass and energy in this limit. The remedy is to include both proper time and rest energy non-relativistically. This means generalizing the meaning of proper time beyond its classical meaning, and introducing the mass as its conjugate momentum. The result is an uncertainty principle between proper time and mass that is very general, and an integral role for both concepts as operators in non-relativistic physics.

Key words: Galilean Transformation; Symmetry; Superselection Rules; Proper Time; Mass.

Introduction

It is a generally accepted rule in non-relativistic quantum theory that one cannot coherently superpose particles of different masses. This rule comes from a demonstration by Bargman [1, 2] that, if one makes a series of transformations to moving coordinate systems, using the Galilean transformation, and eventually arrives back at the original system, one will have produced a phase shift between the components of the wave function representing different mass states. However, the argument goes, these transformations are unphysical. How can merely looking at a wave function from a different coordinate system possibly induce a physically meaningful phase shift that could be detected in an interference experiment? In order to eliminate this possibility, one imposes the superselection rule that one cannot superpose wave functions of different masses.

But there is a very puzzling feature to this result. Relativistically one *can* coherently combine wave functions of different mass states, and the relevant transformation here is the Lorentz transformation. In the non-relativistic limit this reduces to the Galilean transformation, but the phase shift does not disappear in this limit. How can a meaningful relativistic effect that has a non-zero non-relativistic limit be eliminated by a superselection rule, and why should it? Shouldn't instead this non-relativistic limit have a physical interpretation?

We show that from another point of view it is necessary to be able to superpose different mass states non-relativistically. Then we show that there is indeed a physical interpretation to the phase shift described above, namely that it is the “twin paradox” effect [3], the residue of the different proper times elapsed between the original inertial system, and the set of accelerating systems used to describe the particle. So these two sets of coordinate systems are not physically equivalent. The amazing thing is that the difference in proper times between them produces a residue in the non-relativistic limit. This is a phase effect and would not be seen in classical mechanics, but it shows that there are non-relativistic effects due to proper time in quantum theory, and a correct treatment of non-relativistic quantum theory must include the concept of proper time, and the equivalence of mass and energy. The superselection rule prohibiting the superposition of different mass states is inconsistent with the non-relativistic limit of the Lorentz transformation.

Necessity of Non-Relativistic Mass Superpositions

Consider the case of a particle of mass M at rest in an inertial system S decaying into two particles of mass m , flying off in opposite directions (along the x axis) at speed v . Non-relativistically, since mass is conserved,

0932-0784 / 01 / 0100-0067 \$ 06.00 © Verlag der Zeitschrift für Naturforschung, Tübingen · www.znaturforsch.com



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

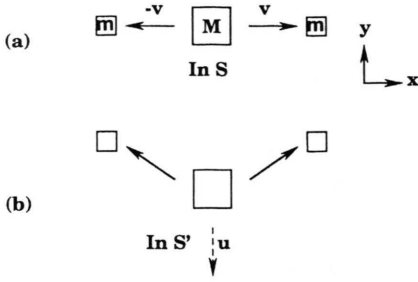


Fig. 1. Decay of a particle into two equal daughter particles. (a) A particle of mass M decays into two equal particles of mass m , seen from a system S where M is at rest. Non-relativistically, since mass is conserved, $M = 2m$. The internal energy ε supplies the velocity v of the daughters, and ε is of order (v^2/c^2) , as is the relativistic mass change. (b) The same decay, as seen from a system S' , moving along the $-y$ axis at speed u .

one would have (see Figure 1a).

$$\begin{aligned} M &= 2m, \\ \varepsilon &= mv^2. \end{aligned} \quad (1)$$

Here, ε is the internal energy of the particle M . Non-relativistically, the mass and energy of the particle are conserved separately. Relativistically,

$$\begin{aligned} M &= 2m\gamma, \\ \gamma &= (1 - v^2/c^2)^{-1/2}, \\ M &= 2m + \varepsilon/c^2. \end{aligned} \quad (2a, b, c)$$

There is no conflict here since relativistically,

$$M = 2m + O(v^2/c^2). \quad (3)$$

Now consider the same decay from a system S' , moving downward along the $-y$ axis at velocity u (see Figure 1b). Here, looking at momentum conservation (non-relativistically) along the y axis,

$$Mu = 2mu. \quad (4)$$

This is perfectly consistent with (1). Relativistically, using $x_\mu = (x, y, z, ict)$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_u & 0 & -i\gamma_u u/c \\ 0 & 0 & 1 & 0 \\ 0 & i\gamma_u u/c & 0 & \gamma_u \end{bmatrix} \begin{bmatrix} v\gamma_v \\ 0 \\ 0 \\ ic\gamma_v \end{bmatrix} = \begin{bmatrix} v\gamma_v \\ u\gamma_u\gamma_v \\ 0 \\ ic\gamma_u\gamma_v \end{bmatrix}, \quad (5)$$

and momentum and energy conservation in the y direction gives

$$\begin{aligned} Mu\gamma_u &= 2mu\gamma_u\gamma_v, \\ Mc\gamma_u &= 2mc\gamma_u\gamma_v. \end{aligned} \quad (6a, b)$$

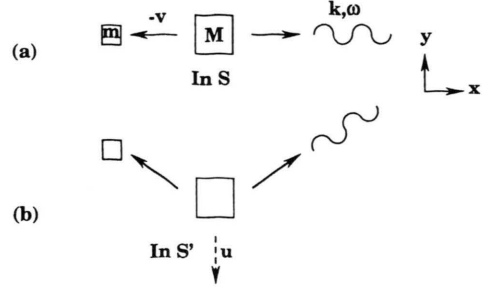


Fig. 2. Decay of an excited particle into its ground state plus a photon. (a) The particle of mass M decays into one of mass m plus a photon. The internal energy ε supplies the recoil and photon energies. Here, ε and the relativistic mass change are of order v/c . (b) The same decay, as seen from the system S' , as before.

Numerically, both of these give the same information as (2a, b, c). However, (6a) says that what looked like energy in the rest system S (in (2a)), looks like inertial mass in the moving system S' , and is essentially one of Einstein's early derivations of, and is the true meaning of, the equivalence of mass and energy. Again, because of (3), there are no surprises here.

However, the situation is very different if one considers M to be the mass of an atom in an excited state, and it decays to its ground state with mass m , emitting a photon of frequency ω (see Figure 2a). Classically, photons do not exist, but this is a situation that one frequently treats non-relativistically in quantum mechanics. In this case, non-relativistic momentum conservation along the x -direction in the S system gives

$$mv = \hbar k. \quad (7)$$

In the S' system momentum conservation in the y direction gives (see Figure 2b)

$$Mu = mu + \hbar k u/c, \quad (8)$$

because both particle and the photon carry momentum in the y direction. But this gives the relation

$$M = m + \hbar\omega/c^2, \quad (9)$$

which, together with (7), gives

$$M = m(1 + v/c), \quad (10)$$

a first order v/c effect, in direct contrast to (3). So even non-relativistically, one cannot ignore the increase in mass, as well as in energy, of the excited state.

Relativistically, (5) gives for the photon, (11)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_u & 0 & -i\gamma_u u/c \\ 0 & 0 & 1 & 0 \\ 0 & i\gamma_u u/c & 0 & \gamma_u \end{bmatrix} \begin{bmatrix} k \\ 0 \\ 0 \\ i\omega/c \end{bmatrix} = \begin{bmatrix} k \\ ku\gamma_u/c \\ 0 \\ i\omega\gamma_u/c \end{bmatrix}$$

and relativistic x -momentum and energy conservation in the system S becomes

$$\begin{aligned} mv\gamma_v &= \hbar k, \\ Mc &= mc\gamma_v + \hbar\omega/c, \end{aligned} \quad (12)$$

while in the S' system, y -momentum and energy become

$$\begin{aligned} Mu\gamma_u &= mu\gamma_u\gamma_v + \hbar ku\gamma_u/c, \\ Mc\gamma_u &= mc\gamma_u\gamma_v + \hbar\omega\gamma_u/c. \end{aligned} \quad (13)$$

Here it is very clear that energy in the S system becomes inertial mass in the S' system, and that the increase of mass in the excited state is necessary for the consistency of the theory. Non-relativistically, (12) reduces to (9).

Quantum mechanically there are many situations where one has coherent combinations of the excited and ground state, which are generally thought of as superpositions of different energy states. But it is clear that for the non-relativistic Galilean transformation to be consistent with the Lorentz transformation, they must also be considered as superpositions of different mass states.

The Quantum-Mechanical Galilean Transformation

The Galilean transformation represents the transformation to a system moving at constant velocity. One can get extra insight into the workings of the Galilean transformation by examining the “extended” Galilean transformation to a rigid system having an arbitrary time-dependent acceleration. The extended Galilean transformation [4] is given by

$$\begin{aligned} r' &= r - \xi(t), \\ t' &= t. \end{aligned} \quad (14)$$

This leads to

$$\begin{aligned} \nabla &= \nabla', \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} - \dot{\xi} \cdot \nabla' \end{aligned} \quad (15)$$

which, when applied to the Schrödinger equation yields

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{\partial \psi}{\partial t} \rightarrow$$

$$-\frac{\hbar^2}{2m} \nabla'^2 \psi = i\hbar \left(\frac{\partial}{\partial t'} - \dot{\xi} \cdot \nabla' \right) \psi. \quad (16)$$

Here we have replaced t' by t . One can eliminate the unwanted $\nabla' \psi$ term by the substitution

$$\psi(r, t) = e^{i\phi(r', t)} \varphi(r', t). \quad (17)$$

Then,

$$\begin{aligned} \nabla' \psi &= (\nabla' \varphi + i \nabla' \phi) e^{i\phi}, \\ \nabla'^2 \psi &= (\nabla'^2 \varphi + 2i \nabla' \phi \cdot \nabla' \varphi + \varphi \nabla'^2 \phi \\ &\quad + \phi (\nabla' \phi)^2) e^{i\phi}, \\ \dot{\psi} &= (\dot{\phi} + i \dot{\phi} \phi) e^{i\phi}, \end{aligned} \quad (18)$$

and the Schrödinger equation becomes

$$\begin{aligned} &-\frac{\hbar^2}{2m} (\nabla'^2 \varphi + 2i \nabla' \phi \cdot \nabla' \varphi + i \phi \nabla'^2 \phi - (\nabla' \phi)^2 \varphi) \\ &= i\hbar \left[(\dot{\phi} + i \dot{\phi} \phi) - \dot{\xi} \cdot (\nabla' \varphi + i \phi \nabla' \phi) \right]. \end{aligned} \quad (19)$$

One can choose ϕ such as to eliminate the terms in $\nabla' \phi$, which gives $\phi = \frac{m}{\hbar} \dot{\xi} \cdot r' + g(t)$. Then one can choose $g(t)$ such as to eliminate the purely time-dependent terms, and one finally arrives at

$$\begin{aligned} \phi &= \frac{m}{\hbar} \left(\dot{\xi} \cdot r' + \frac{1}{2} \int \dot{\xi}^2 dt \right), \\ &-\frac{\hbar^2}{2m} \nabla'^2 \varphi + m \ddot{\xi} \cdot r' \varphi = i\hbar \dot{\phi} \varphi, \\ \psi(r, t) &= e^{i \frac{m}{\hbar} \left(\dot{\xi} \cdot r' + \frac{1}{2} \int \dot{\xi}^2 dt \right)} \varphi(r', t). \end{aligned} \quad (20)$$

This form of the Schrödinger equation shows that in the accelerated system there appears a gravitational field, and so this is the expression of the strong equivalence principle in quantum theory. However, it can also be used to show another facet of the Galilean transformation, because the phase factor has a strong physical interpretation.

Assume that there exists a superposition of two different masses, m_1 and m_2 , so that the wave function can be written in an inertial system S as

$$\psi = \psi_1(m_1, r, t) + \psi_2(m_2, r, t). \quad (21)$$

Then assume that one can describe the same superposition in an accelerating system S' that obeys (14), with $\xi = \xi(t)$, $\xi(0) = \xi(T) = 0$, so that the system S' performs a closed circuit and coincides with the system S at times $t = 0$ and $t = T$, such that $r'(T) = r(T)$. However, according to (20) one can write in S , where

$\xi(t) = 0$:

$$\psi_s(T) = \varphi_1 + \varphi_2. \quad (22)$$

while in S' :

$$\begin{aligned} \psi_{S'} &= e^{i \frac{m_1}{2\hbar} \int_0^T \xi^2 dt} \varphi_1 + e^{i \frac{m_2}{2\hbar} \int_0^T \xi^2 dt} \varphi_2 \\ &= e^{i \frac{m_1}{2\hbar} \int_0^T \xi^2 dt} (\varphi_1 + e^{-i \frac{\Delta m}{2\hbar} \int_0^T \xi^2 dt} \varphi_2), \\ \Delta m &= m_1 - m_2. \end{aligned} \quad (23)$$

At the time T and beyond, S and S' are the same system. One has made a transformation to an accelerating system, which has returned to the original system at time T . Thus one has described the same physical system, ψ , in two different coordinate systems, and yet the second system has induced a relative phase shift between the two components φ_1 and φ_2 , relative to the first system. This phase shift would be detectable in an interference experiment, but it has no physical significance, merely relating to how one would describe the same state in a different coordinate system.

Bargmann introduced the way around this dilemma, that has been used ever since, namely to require that $\Delta m = 0$. Thus, in order to eliminate the unphysical phase shift, one places a superselection rule on the system and requires that particles of different masses cannot be superposed in non-relativistic quantum mechanics. (Bargmann actually performed a series of translations and ordinary constant velocity Galilean transformations, but we shall show in the appendix that our subsequent argument holds in that case also.)

The Inconsistency of the Superselection Rule

We believe that the above solution, that masses cannot be superposed non-relativistically, is inconsistent with the principles of quantum mechanics and relativity. Not only does it contradict the result we previously arrived at, but it does so for a very definite reason. It is just not true that the difference between the two systems S and S' in the previous section is unphysical. In fact, if one were attached to the system S' while it underwent its acceleration, one's clocks would be running at a different rate than those in the system S , and when one arrived back in S at time T , less time would have passed in the system S' than in the system S . This effect is indeed just the standard twin paradox of special relativity. If one closed one's eyes at time $t = 0$, and opened them at time T , and were asked which system is the one that had ac-

celerated, while one could give no answer in classical physics, this is not true in special relativity. There, one would say that the system for which less time had elapsed is the system that has been accelerated. And in fact, in relativistic quantum mechanics the generalization of the Galilean transformation is the Lorentz transformation, and relativistically one can superpose different masses. The effect is real, and it leaves a residue in the non-relativistic limit. And in fact, the difference in proper times between the two coordinate systems, S and S' , is

$$\tau_1 - \tau_2 = t - \int \sqrt{1 - \xi^2/c^2} dt \rightarrow \frac{1}{2c^2} \int \xi^2 dt \quad (24)$$

in the non-relativistic limit.

The space-time phase factor of a plane wave to an observer moving with the particle

$$\begin{aligned} k \cdot r - \omega t &= \frac{mv\gamma t}{\hbar} \cdot vt - \frac{mc^2\gamma}{\hbar} t \\ &= -\frac{mc^2\gamma}{\hbar\gamma^2} t = -\frac{mc^2\tau}{\hbar}, \end{aligned} \quad (25)$$

is an invariant and holds for arbitrary motion. In the systems S and S' we have

$$\begin{aligned} \psi_S &= e^{-im_1 c^2 \tau_1 / \hbar} \varphi_1 + e^{-im_2 c^2 \tau_1 / \hbar} \varphi_2 \\ &= e^{-im_1 c^2 \tau_1 / \hbar} (\varphi_1 + e^{i\Delta m c^2 \tau_1 / \hbar} \varphi_2) \\ &= e^{-im_1 c^2 \tau_1 / \hbar} (\varphi_1 + \varphi'_2), \\ \psi_{S'} &= e^{-im_1 c^2 \tau_2 / \hbar} \varphi_1 + e^{-im_2 c^2 \tau_2 / \hbar} \varphi_2 \\ &= e^{-im_1 c^2 \tau_2 / \hbar} (\varphi_1 + e^{i\Delta m c^2 \tau_2 / \hbar} \varphi_2) \\ &= e^{-im_1 c^2 \tau_2 / \hbar} (\varphi_1 + e^{-i\Delta m c^2 \Delta \tau / \hbar} \varphi'_2), \end{aligned} \quad (26)$$

where $\varphi'_2 = e^{i\Delta m c^2 \tau_1 / \hbar} \varphi_2$, and we see that the extra phase shift in the non-relativistic limit is

$$e^{-i\Delta m c^2 \Delta \tau / \hbar} \rightarrow e^{-\frac{i\Delta m}{2\hbar} \int \xi^2 dt}, \quad (27)$$

which agrees with that of (23).

Relativistically, the phase factor $e^{-mc^2 \tau / \hbar}$ is of course an invariant under a Lorentz transformation, but in deriving the non-relativistic Schrödinger equation this invariance is destroyed by factoring out the time dependence $e^{-mc^2 t / \hbar}$. The phase that remains, and which shows up explicitly in the Galilean transformation, is $e^{-mc^2(\tau - t) / \hbar}$. This phase, which is independent of c , and which therefore shows up in the non-relativistic limit, is a real effect, and it leads to the phase of (23). The problem is that although real, it is uninterpretable in the non-relativistic limit, where proper time is not recognized.

The standard answer has been to eliminate this phase effect by fiat, with the creation of a superselection rule, disallowing the appearance of superposition of differing masses. But this is inconsistent because these superpositions do occur relativistically, and they do cause measurable phase shifts which persist in the non-relativistic limit. There is no point, or validity, in trying to eliminate the phase shift non-relativistically, because it is real and has a physical interpretation.

In classical physics this type of problem doesn't come up, as it is a phase problem, but in quantum mechanics we have this non-local integral over all past times that keeps track of the proper time difference, and it is not legitimate to ignore it, or to legislate it away, since in fact it can produce real interference effects. The correct way around this problem is to concede that quantum-mechanically we must keep track of rest-mass and proper time differences when they appear as non-vanishing phases in the non-relativistic limit and learn to incorporate them into the theory.

It should not come as such a surprise that there are residual phase shifts due to proper time that persist in the non-relativistic limit, since the non-relativistic Lagrangian itself is the residuum of such a relativistic effect. In the non-relativistic limit we have

$$\begin{aligned}
 -mc^2 (d\tau - dt) &= -mc^2 \left(\sqrt{g_{\mu\nu} dx^\mu dx^\nu} - dt \right) \\
 &\rightarrow -mc^2 \left(\sqrt{g_{00} - \left(\frac{dx^i}{dt} \right)^2} - 1 \right) dt \\
 &\rightarrow -mc^2 \left(\sqrt{1 + \frac{2\phi}{c^2} - \frac{v^2}{c^2}} - 1 \right) dt \\
 &\rightarrow \left(-mc^2 + \frac{1}{2}mv^2 - m\phi + mc^2 \right) dt \\
 &\rightarrow (T - V) dt = L dt. \tag{28}
 \end{aligned}$$

These are the effects that contribute to the non-relativistic Feynman path integral, and in special circumstances can cause other quantum residual effects.

Proper Time as a Physical Variable and Operator

One is normally used to interpreting proper time kinematically as $\tau = \int (1 - v^2/c^2)^{1/2} dt$ along the trajectory of the particle. However this is clearly a classical interpretation, as it presumes a particular trajectory for the particle, an idea inconsistent with quantum mechanics. What one needs is an operator τ that can be interpreted at every point (x, t) in configuration space. And just as

it is only in the classical limit that x defines a trajectory, by virtue of an equation of motion, given by Hamilton's equations

$$v = \frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}, \tag{29}$$

where p is the momentum conjugate to x , so too the formalism should be able to extend itself such that the same thing happens with the proper time. It turns out that this extension is very natural [5]. The mass (times c^2) is the conjugate momentum to the proper time, and the Hamiltonian then becomes $H = H(x, p; \tau, m)$. The extra equations of motion are then

$$\frac{d\tau}{dt} = \frac{1}{c^2} \frac{\partial H}{\partial m}, \quad c^2 \frac{dm}{dt} = -\frac{\partial H}{\partial \tau}. \tag{30}$$

As an example, for a free particle the Hamiltonian and equations of motion would be

$$\begin{aligned}
 H &= \sqrt{p^2 c^2 + m^2 c^4} = E, \\
 v &= \frac{\partial H}{\partial p} = \frac{p}{E}, \quad \frac{d\tau}{dt} = \frac{1}{c^2} \frac{\partial H}{\partial m} = \frac{mc^2}{E}, \\
 \frac{dp}{dt} &= \frac{dm}{dt} = 0; \\
 p &= mv / \sqrt{(1 - v^2/c^2)}, \\
 E &= mc^2 / \sqrt{(1 - v^2/c^2)}, \\
 \frac{d\tau}{dt} &= \sqrt{(1 - v^2/c^2)}, \tag{31}
 \end{aligned}$$

where the last three lines come from inverting the equation for $v = v(p)$. One sees here that the equation for $\tau(t)$ appears as an equation of motion and is no longer a kinematic identity. The Hamiltonian formalism also provides the operator definition of m , which is parallel to that of p , namely

$$p_{\text{op}} = \frac{\hbar}{i} \frac{\partial}{\partial x}, \quad m_{\text{op}} = \frac{1}{c^2} \frac{\hbar}{i} \frac{\partial}{\partial \tau}. \tag{32}$$

How can one quantum-mechanically interpret τ as an operator? The Galilean transformation provides the key to how this should be done. From (20) one has

$$\begin{aligned}
 \psi(x, t) &= e^{im(\dot{\xi} \cdot dr' + \dot{\xi}^2 dt/2)/\hbar} \phi(r', t' = \tau), \\
 dr' &= dr - \dot{\xi} dt, \\
 \psi(x, t) &= e^{im(\dot{\xi} \cdot dr - \dot{\xi}^2 dt/2)/\hbar} \phi(r', \tau) \\
 &= e^{[(\dot{\xi} \cdot dr - \dot{\xi}^2 dt/2)/c^2] \partial/\partial \tau} \phi(r', \tau) \\
 &= \phi\left(r', \tau + \frac{1}{c^2} (\dot{\xi} \cdot dr - \dot{\xi}^2 dt/2)\right). \tag{33}
 \end{aligned}$$

Here we have interpreted the time passed in the accelerated frame as the proper time, measured in that frame. The time t is the laboratory time. Also, we have used the fact that the mass operator acts as a translation operator in τ . One then sees that this is merely the Lorentz transformation expressing itself non-relativistically,

$$d\tau = \frac{1}{\sqrt{1 - \dot{\xi}^2/c^2}} (dt - \dot{\xi} \cdot dr/c^2) \\ \xrightarrow{\text{NR}} dt + (\dot{\xi}^2 dt/2 - \dot{\xi} \cdot dr)/c^2. \quad (34)$$

This yields the interpretation of τ . If the particle is located at some point (r_1, t_1) at some time t_1 , say by passing through a slit, and a counter is placed at (r_2, t_2, ξ) , moving with velocity $\dot{\xi}$, then the proper time passage is given by (34), where $dr = r_2 - r_1$, and $dt = t_2 - t_1$. By altering the velocity $\dot{\xi}$, one can alter the value of $d\tau$, and thus one has control over the value of τ , and it can be defined at any r and t , and not just over a classical trajectory. One can define it along the trajectory of the particle by locating the counter on the trajectory, moving with the velocity of the particle. Thus $dr = \dot{\xi} dt$, where $\dot{\xi} = v$, the velocity of the particle. Similarly, if the counter is at rest, as is usually the case, then one has $\tau = t$, and there is no information to be gained from consideration of the proper time. But in general one can define it along any path.

The Mass as a Physical Variable and Operator

If one thinks of the mass as an operator, as given by (32), then the Klein-Gordon equation for a free particle in one dimension becomes

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial \tau^2} + \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0. \quad (35)$$

One can change the variables to t' and u , given by

$$t' = t, \quad u = \tau - t, \quad (36)$$

and the derivatives take the form

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \frac{\partial}{\partial u}. \quad (37)$$

Then the Klein-Gordon equation becomes

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u} \right)^2 \psi \\ = \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} + \frac{2}{c^2} \frac{\partial^2 \psi}{\partial t \partial u} = 0. \quad (38)$$

In order to approach the non-relativistic limit, write

$$\psi = e^{i(mc^2/\hbar)u} \varphi \quad (39)$$

and drop the second time derivative term, which is down by (v^2/c^2) . Then φ obeys the non-relativistic Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2} = i\hbar \frac{\partial \varphi}{\partial t}. \quad (40)$$

For a detector at rest, the solution is the standard one, as the phase factor in u disappears. But for a moving detector, the extra phase can have the same order of magnitude as the phase of the solution φ . The extra phase factor in u corresponds to the $e^{-imc^2 t/\hbar}$ "zitterbewegung" term factored out in the conventional solution. So in our case there is essentially no relativistic zitterbewegung, but there can be a non-relativistic residue, not there in the usual theory. This answers a problem that has always bothered me, namely how it is possible that the very rapidly oscillating zitterbewegung, which completely dominates the non-relativistic contribution, could perfectly cancel out, even for widely separated components of the wave function.

Relativistically, the plane wave function solution of (35) is

$$\psi = A e^{i(mc^2 \tau/\hbar + kx - \omega t)}, \quad (41)$$

where k and ω have their relativistic values. For a detector moving along with the particle, the phase factor is equal to 0. Note that the mass in this equation does not have to be the rest mass of the particle because of the freedom given by the Klein-Gordon equation (35), where the mass is an operator. We shall see in the next section that it can contain contributions from binding energies and energy uncertainties.

Just as the phase factor in k and x gives rise to the uncertainty principle in p and x , so in the non-relativistic limit the phase factor in (39) gives rise to an uncertainty principle between $u = \tau - t$, and m , of the form

$$c^2 \Delta m \cdot \Delta u \geq \hbar/2. \quad (42)$$

There are many examples of this uncertainty relation [6, 7], and it is very general. If one wants to measure the mass of a particle to within Δm , then the proper time on a clock sitting on the particle will be unknown to within Δu .

In order to impart the flavor of this relation, we will consider just one example. Imagine a charged particle, whose velocity v_0 is accurately known, passes through a slit of width d , beyond which is an electric field per-

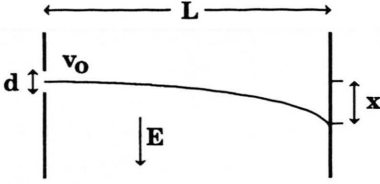


Fig. 3. The mass-proper time uncertainty relation: A particle of mass m , with speed v_0 , passes through a slit of width d and is deflected by an electric field perpendicular to its initial motion. It travels a time $T = L/v_0$, and is displaced by a distance x . A measurement of x will give the value of m . A clock sitting on the particle will give the passage of its proper time τ . But the spread in transverse velocity due to diffraction at the incident slit prevents an accurate measure of τ , even if T is known accurately. This spread in τ is correlated through v_0 with the uncertainty in x , and therefore with that in m .

pendicular to the path of the particle (see Figure 3). The particle is collected on a screen a distance L away, and its mass is to be determined by its total deflection x . We have

$$x = \frac{eE}{2m} T^2, \quad m = \frac{eE}{2x} T^2, \quad \frac{\Delta m}{m} = \frac{\Delta x}{x} \geq \frac{d}{x}. \quad (43)$$

Here we assume that $T = L/v_0$ can be accurately measured, and that the irreducible error in x is the size of the slit, as we cannot know where within the slit the particle passed.

The passage of proper time along the path of the particle is given by

$$u = \tau - T = \frac{1}{2} \int \frac{v^2}{c^2} dt. \quad (44)$$

Even if T is known very accurately, v is affected by the angular spread of the particle as it passes through the slit. We have

$$\begin{aligned} v_x &= v_{0x} + \alpha t, \quad \alpha = \frac{eE}{m}, \\ u &= \frac{1}{2c^2} \int [v_0^2 + (v_{0x} + \alpha t)^2] dt, \\ \Delta u &\approx \frac{1}{2c^2} \int 2 \Delta v_{0x} \alpha t dt = \frac{\Delta v_{0x} \alpha T^2}{2c^2} = \frac{x \Delta v_{0x}}{c^2}. \end{aligned} \quad (45)$$

Here we have used the fact that the largest error in u will be induced by the cross term in v_x , since $\langle v_{0x} \rangle = 0$.

For a small angle of diffraction

$$\begin{aligned} \Delta v_{0x} &= v_0 \theta = v_0 \lambda / d = v_0 h / p d = h / m d, \\ \Delta u &= \frac{x \Delta v_{0x}}{c^2} = \frac{x h}{c^2 m d}, \\ c^2 \Delta m \cdot \Delta u &\approx c^2 \left(\frac{m d}{x} \right) \left(\frac{x h}{c^2 m d} \right) = h. \end{aligned} \quad (46)$$

In this example, the m, u uncertainty relation has been reduced to the p_x, x uncertainty relation, and in general this kind of phenomenon happens. So even if the time in the laboratory can be measured very accurately, the time passed on a clock moving with the particle cannot.

The Quantum-Mechanical Meaning of Mass

There is one further very important point that is brought out by this example, as well as many other examples, and that is that relativistically, and quantum-mechanically, the mass of a particle is *not* the rest mass of the free particle. So even if one knows, say, that the particle is a proton, one cannot say that $\Delta m = 0$. This may be true classically, but quantum mechanically it is not. The mass is defined as the energy of the particle in the rest system, and quantum mechanically this generally includes an uncertainty in the energy. If the particle or system is moving, we can say that the mass is the energy in the barycentric system, that is the system with zero momentum. This definition includes any binding energy the particle may have, or any spread in energy. So the rest energy of the particle may be called the “nominal” mass, but the inertial mass of the system, as we saw earlier, includes these other internal energies and energy uncertainties.

For a particle, or a system of particles, we can write

$$\bar{P} = \bar{V} \bar{E} / c^2, \quad (47)$$

where the bars indicate expectation values. The symbol \bar{V} defines the average velocity of the system. One can use this equation to define a velocity operator, V_{op} , from the equation

$$\frac{\hbar}{i} \nabla \psi(r, t) = V_{\text{op}} \frac{i\hbar}{c^2} \frac{\partial \psi}{\partial t}, \quad (48)$$

$$V_{\text{op}} \psi = c^2 \sum \frac{P_n}{E_n} a_n \varphi_n \xrightarrow{\text{NR}} c^2 \sum \frac{P_n}{m_0} a_n \varphi_n,$$

where $\psi = \sum a_n \varphi_n(r) e^{-iE_n t / \hbar}$, an expansion in the simultaneous eigenfunctions of P and E (defined by the time derivative, and not the Hamiltonian operator.) Then the mass operator can be defined as the energy in the barycentric system, $P_B = 0$, from a Lorentz transformation,

$$\begin{aligned} P' &= P_B = \gamma_{\bar{V}} (P - \bar{V} E / c^2), \quad \langle P_B \rangle = 0, \\ E' &= E_B = M_{\text{op}} = \gamma_{\bar{V}} (E - \bar{V} P), \quad \langle E_B \rangle = \bar{M}, \\ \bar{M} &= \sqrt{1 - \bar{V}^2 / c^2} \bar{E}. \end{aligned} \quad (49)$$

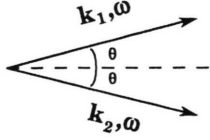


Fig. 4. The mass of a photon: A non-collinear two-photon system has a barycentric system traveling with a speed less than c , and in this system, the energy is the rest energy of the system, and therefore its mass. Similarly, if this were a one photon system, with equal amplitudes for being in either of the two photon states, its momentum, energy expectation values will behave in the same way, and act like a massive particle, even though each of its component amplitudes has zero mass.

These equations can give surprising results. For example it is well known that a two-photon system, where the two photons are not collinear, has a barycentric system. Take the two photons in Fig. 4, where the photons each have the same ω , but are separated by an angle 2θ . This system has a total momentum $P = 2\hbar k \cos\theta$, and energy $E = 2\hbar\omega$. Therefore the mass and velocity are, $\bar{M}c^2 = 2\hbar\omega \sin\theta$, $\bar{V} = c \cos\theta$. However quantum-mechanically, for a single photon, whose wave function consists of a linear superposition of equal amounts of these two amplitudes, the result is exactly the same (without the factor of 2). This is true even though each separate amplitude has mass 0. This is because, if an ensemble of such photons struck a wall and were absorbed, while each separate hit would act like a massless particle, the expectation value would act the same as though the wall were struck by a massive particle of mass \bar{M} .

Thus, in keeping with the spirit of relativity and quantum mechanics, the mass is the inertial mass of the system and includes all internal energies, such as binding energies, as well as uncertainties in the energy in the rest system [8]. This latter is a problem that doesn't show up classically, but the inclusion of such energy uncertainties is necessary for the consistency of the theory, and so quantum mechanically one cannot merely take the mass as m_0 , the free particle rest mass. Indeed, it would be inconsistent to do so.

Conclusion

We have shown that the superselection rule that masses cannot be coherently combined is inconsistent, and that there are situations in which the concepts of proper time and rest mass enter in the non-relativistic limit. This in turn clears up another controversy that has raged almost since the beginning of quantum mechanics, for

many people have used rest mass and proper time in non-relativistic arguments, and they have often been taken to task that it was inconsistent to do so. Probably the most famous case occurred in Bohr's refutation [9] of Einstein's argument that the weighing of a box of photons violates the $\Delta E \cdot \Delta t$ uncertainty principle. But we have seen that these concepts do enter non-relativistically, and in some circumstances are required to do so. And while it is probably true that the injudicious use of these ideas can cause problems [8], it is also true that they do and should play a role quantum mechanically in the non-relativistic limit.

Another point that ought to be made is that the very idea of a wave function containing a superposition of different mass states implies that the mass is the eigenvalue of some operator. We believe that the moral is that the mass and proper time should be treated as operators in quantum mechanics, an idea that has far-reaching implications.

This research has been supported in parts by NSF Grant PHY-97-22614.

Appendix: Bargmann's Original Proof

In his original proof, Bargmann [1] used translations and the standard Galilean transformation (for constant velocities), rather than the extended transformation, making a series of transformations from the system S , ending back at the original system. The series of transformations he used were from S to S_1 , by making a translation by a ; to S_2 , by making a boost to velocity v ; to S_3 , by making another translation by $-a$; and finally to $S_4 = S$, by making another boost by velocity $-v$, ending up in the original system. Non-relativistically, the times were not affected by these transformations, but spatially,

$$\begin{aligned} r_1 &= r - a, & r_2 &= r_1 - vt, & r_3 &= r_2 + a, \\ r_4 &= r_3 + vt = r. \end{aligned} \quad (A1)$$

The translations follow the simple rule

$$\psi_1(r_1) = \psi(r), \quad \psi_3(r_3) = \psi_2(r_2), \quad (A2)$$

while the boosts follow (20), with $\xi = \pm vt$. Thus

$$\begin{aligned} \psi(r) &= e^{\frac{i}{\hbar}(\pm v \cdot r' + v^2 t/2)} \psi'(r') \\ &= e^{\frac{i}{\hbar}(\pm v \cdot (r \pm vt) + v^2 t/2)} \psi'(r'), \\ \psi'(r') &= e^{\frac{i}{\hbar}(\pm v \cdot r - v^2 t/2)} \psi(r), \end{aligned} \quad (A3)$$

and so

$$\begin{aligned}\psi_4(r_4) &= e^{-i \frac{m}{\hbar} (-v \cdot r_3 - v^2 t/2)} \psi_3(r_3), \\ \psi_3(r_3) &= \psi_2(r_2), \\ \psi_2(r_2) &= e^{-i \frac{m}{\hbar} (v \cdot r_1 - v^2 t/2)} \psi_1(r_1), \\ \psi_1(r_1) &= \psi(r),\end{aligned}\quad (\text{A4})$$

implying

$$\psi_4(r_4) = e^{-i \frac{m}{\hbar} (v \cdot a)} \psi(r). \quad (\text{A5})$$

This in turn imparts to the original wave function of (20), consisting of a superposition of two masses,

$$\psi(t=0) = \psi_I(m_1, r, 0) + \psi_{II}(m_2, r, 0), \quad (\text{A6})$$

the extra phase shift

$$\begin{aligned}\psi(T) &= e^{i \frac{m_1}{\hbar} (va)} \psi_I(r) + e^{i \frac{m_2}{\hbar} (va)} \psi_{II}(r) \\ &= e^{i \frac{m_1}{\hbar} (va)} (\psi_I(r) + e^{-i \frac{\Delta m}{\hbar} (va)} \psi_{II}(r)),\end{aligned}\quad (\text{A7})$$

rather than the phase shift of our transformation, (23). However in this case also, this phase shift is just that due to the difference in proper time elapsed between the two systems, exactly as in our case. To see this note the

transformations relativistically,

$$\begin{aligned}r_1 &= r - a, \quad t_1 = t, \\ r_2 &= \gamma(r_1 - vt_1), \quad t_2 = \gamma(t_1 - vr_1/c^2), \\ r_3 &= r_2 + a, \quad t_3 = t_2, \\ r_4 &= \gamma(r_3 + vt_3), \quad t_4 = \gamma(t_3 + vr_3/c^2); \\ \gamma &= (1 - v^2/c^2)^{-1/2}.\end{aligned}\quad (\text{A8})$$

This gives, in the non-relativistic limit,

$$\begin{aligned}x_4 &= x + (\gamma - 1)a \xrightarrow{\text{NR}} x, \\ t_4 &= t + \gamma va/c^2 \xrightarrow{\text{NR}} t + va/c^2.\end{aligned}\quad (\text{A9})$$

Thus the phase factor of (A7), like that of (26), is just $e^{-i\Delta mc^2\Delta\tau/\hbar}$. It is clear that in this case the proper time difference is due primarily to the lack of simultaneity between the points r and $r - a$. But, as in the case considered earlier, this causes a phase shift that does not disappear, and is independent of c , in the non-relativistic limit. Thus here too, the two coordinate systems, the inertial system S and the accelerated system, are not physically equivalent in that there is a proper time difference between them, a twin-paradox effect, that leaves a non-relativistic residue which is measured by the above phase factor. And so here again, the phase shift is of a physical origin and there is no legitimate reason to apply the superselection rule on mass.

[1] V. Bargmann, Ann. Math. **59**, 1 (1954).

[2] See also J.-M. Lévy-Leblond, in Group Theory and Its Applications, Vol II, E. M. Loebl, ed., Academic Press, New York 1971; J. Math. Phys. **4**, 776 (1963); and H. Bernstein, J. Math. Phys. **8**, 406 (1967). It is also discussed in the text by F. A. Kaempffer, Concepts in Quantum Mechanics, Academic Press, New York 1965.

[3] D. M. Greenberger, Amer. J. Phys. **47**, 35 (1979).

[4] J. Rosen, Amer. J. Phys. **40**, 683 (1972). See also [3].

[5] The outlines of this formalism were worked out in a series of papers: D. M. Greenberger, J. Math. Phys. **11**, 2329 (1970); **11**, 2341 (1970); **15**, 395 (1974); **15**, 406 (1974).

[6] See the papers in [5]. Also, D. M. Greenberger, contribution to Experimental Metaphysics: Quantum Mechanical Studies for Abner Shimony, Vol. I, R. S. Cohen, M. Horne, and J. Stachel, eds., Kluwer Academic, Dordrecht 1997.

[7] A recent paper on this topic: S. Kudaka and S. Matsumoto, J. Math. Phys. **40**, 1237 (1999).

[8] In our example, a two-photon system, or two amplitude one-photon system, has a barycentric system, and carries an equivalent mass proportional to $\Delta E/c^2$, which is much smaller than the energy of the photon(s). This is not at all the same as attributing a mass E/c^2 to a single photon, a dangerous procedure that has been warned against by L. B. Okun, K. G. Selivanov, and V. L. Telegdi, Amer. J. Phys. **68**, 115 (2000).

[9] N. Bohr, in Albert Einstein: Philosopher, Scientist, P. A. Schilpp, ed., Tudor, New York 1951. This book has been recently republished by Dover Press.